The University of Alabama at Birmingham (UAB)
Department of Physics

PH 461/561 – Classical Mechanics I – Fall 2005

<u>Assignment # 8</u> Due: **Thursday, October 20**

1. As studied in class, the damped harmonic oscillator allows three types of solutions:

   <u>Weak damping:</u> $\gamma < \omega_0$: $x(t) = Ae^{-\gamma t}\cos(\omega_d t + \phi)$
   (underdamping)

   <u>Critical damping:</u> $\gamma = \omega_0$: $x(t) = C_1 e^{-\gamma t} + C_2 t e^{-\gamma t}$

   <u>Strong damping:</u> $\gamma > \omega_0$: $x(t) = C_1 e^{-\left(\gamma - \sqrt{\gamma^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\gamma + \sqrt{\gamma^2 - \omega_0^2}\right)t}$
   (overdamping)

   Exponential factors appear in all three solutions and determine the *decay rate* of the motion in each case. An inspection of the above equations reveals that the *<u>decay parameter</u>* that dominates the decrease in amplitude for each case is as follows:

   <u>Weak damping:</u> $\gamma < \omega_0$: $(\text{decay parameter}) = \gamma$
   (underdamping)

   <u>Critical damping:</u> $\gamma = \omega_0$: $(\text{decay parameter}) = \gamma$

   <u>Strong damping:</u> $\gamma > \omega_0$: $(\text{decay parameter}) = \gamma - \sqrt{\gamma^2 - \omega_0^2}$
   (overdamping)

   Note: In the case of strong damping, the decay parameter is chosen as the smallest of the two decay rates, because it dominates the decay for large *t*.

   a) For fixed ω_0, sketch the behavior of the *<u>decay parameter</u>* as a function of γ for $0 < \gamma < \infty$.

      Your sketch should:

      i. Verify that the decay parameter for an overdamped oscillator *decreases* with increasing γ.

      ii. Indicate the value of γ for which the decay parameter is maximum.

   b) Explain the meaning of the maximum in the value of the decay parameter.

2. Verify that the function $x(t) = te^{-\gamma t}$, is indeed a second solution of the equation of motion for a critically damped oscillator ($\gamma = \omega_0$)

3. Find the rate of change of the energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ for a damped oscillator and show that the dE/dt is (minus) the rate at which energy is dissipated by the damping force $-b\dot{x}$.

4. A mass *m* subject to a linear restoring force $-kx$ and damping $-b\dot{x}$ is displaced a distance x_0 from equilibrium and released with zero initial velocity. Find the motion in the underdamped, critically damped, and overdamped cases.

5. Solve Problem 4 for the case when the mass starts from its equilibrium position with an initial velocity v_0. Sketch the motion for the three cases.

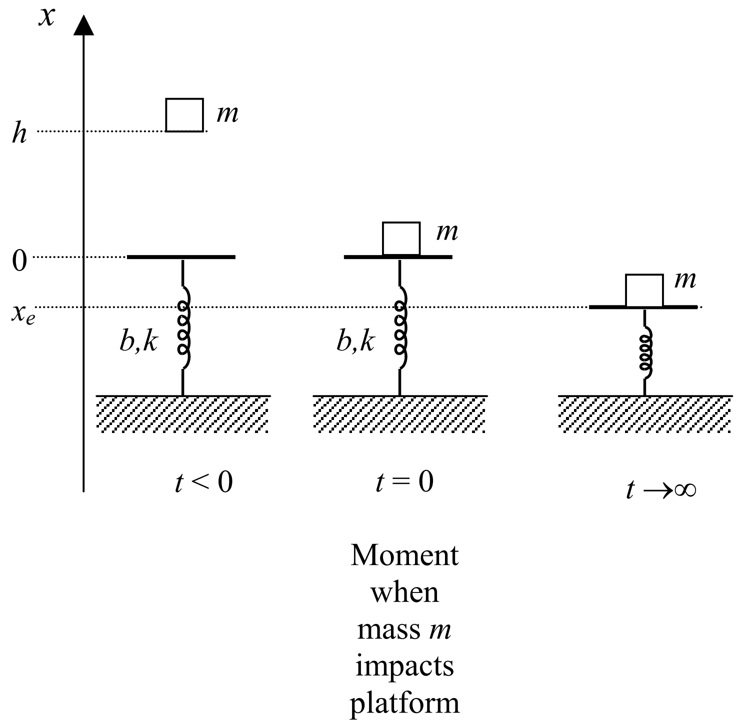
6. Solve Problem 4 for the case when the mass has an initial displacement x_0 and initial velocity v_0 directed toward the equilibrium point. Show that for a large enough value of v_0 (namely if $|v_0| > |(\gamma + \beta)x_0|$; where $\beta = \sqrt{\gamma^2 - \omega_0^2}$), the mass will overshoot the equilibrium in the critically damped and overdamped cases.

   Sketch the motion in these cases.

7. A mass of 1000 kg falls from a height of 10 m over a platform of negligible mass. One is interested in designing a spring/shock absorber system on which the platform will be mounted, such that the platform will reach a new equilibrium position 0.2 m below its original position as quickly as possible after the impact and without going beyond it (See Figure in the next page).

   a. Find the spring constant *k* and the damping constant *b* of the shock absorber. ake sure the solution *x*(*t*) found satisfies the correct initial conditions and that the platform does not go beyond the new position of equilibrium. (i.e., ensure there is no overshooting).

   b. Determine, up to two significant digits, the time it takes for the platform to position itself within 1 mm of its final position.



8. **(a)** Find an expression for the phase space trajectories of the free harmonic oscillator. **(b)** Sketch the trajectories for various values of the total energy of the oscillator. **(c)** Discuss whether the trajectories are open or closed and explain the significance of this fact.

9. **(a)** Find an expression for the phase space trajectories of an *underdamped* harmonic oscillator. **(b)** Sketch the trajectories for various values of the total energy of the oscillator. **(c)** Discuss whether the trajectories are open or closed and explain the significance of this fact in relation to the periodicity of the motion. **(d)** Show that the phase space trajectories you found agree with the behavior of $x(t)$ and $\dot{x}(t)$ for the underdamped oscillator.

10. A free harmonic oscillator, initially at rest, is subject beginning at $t = 0$ to an applied force $F_0 \sin \omega t$.

    a. Find the motion $x(t)$.
    b. Discuss the physical meaning of the solution $x(t)$.
    c. Find an expression for $x(t)$ in the limit of exact resonance (i.e., $\omega \to \omega_0$) and show that the amplitude of the oscillation increases linearly with time. Sketch $x(t)$ in this case.